Overlap Fluctuations from Random Overlap Structures

Adriano Barra[*], Luca De Sanctis [†]

February 6, 2008

Abstract

We investigate overlap fluctuations of the Sherrington-Kirkpatrick mean field spin glass model in the framework of the Random Overlap Structure (ROSt). The concept of ROSt has been introduced recently by Aizenman and coworkers, who developed a variational approach to the Sherrington-Kirkpatrick model. We propose here an iterative procedure to show that, in the so-called Boltzmann ROSt, Aizenman-Contucci (AC) polynomials naturally arise for almost all values of the inverse temperature (not in average over some interval only). The same results can be obtained in any Quasi-Stationary ROSt, including therefore the Parisi structure. The AC polynomials impose restrictions on the overlap fluctuations in agreement with Parisi theory.

[*]King's College London, Department of Mathematics, Strand, London WC2R 2LS, United Kingdom, and Dipartimento di Fisica, Università di Roma "La Sapienza" Piazzale Aldo Moro 2, 00185 Roma, Italy, <Adriano.Barra@roma1.infn.it>

[†]ICTP, Strada Costiera 11, 34014 Trieste, Italy, <lde_sanc@ictp.it>

1 Introduction

The study of mean field spin glasses has been very challenging from both a physical and a mathematical point of view. It took several years after the main model (the Sherrington-Kirkpatrick, or simply SK) was introduced before Giorgio Parisi was able to compute the free energy so ingeniously ([14] and references therein), and it took much longer still until a fully rigorous proof of Parisi's formula was found [13, 16]. Parisi went beyond the solution for the free energy and gave an Ansatz about the pure states of the model as well, prescribing the so-called ultrametric or hierarchical organization of the phases ([14] and references therein). From a rigorous point of view, the closest the community could get so far to ultrametricity are identities constraining the probability distribution of the *overlaps*, namely the Aizenman-Contucci (AC) and the Ghirlanda-Guerra identities (see [1, 11] respectively). For further reading, we refer to [9, 8, 15], but also to the general references [17, 7]. Most of the few important rigorous results about mean field spin glasses can be elegantly summarized within a powerful and physically profound approach introduced recently by Aizenman et al. in [2]. We want to show here that in this framework the AC identities can be deduced too. This is achieved by studying a stochastic stability of some kind, similarly to what is discussed in [8], inside the environment (the *Random Overlap Structure*) suggested in [2], and taking into account also the intensive nature of the internal energy density. A central point of the treatment is a power series expansion similar to the one performed in [5].

The paper is organized as follows. In section 2 we introduce the concept of Random Overlap Structure (henceforth ROSt), and use it to state the

related Extended Variational Principle. In section 3 we present the main results regarding the AC identities and similar families of relations. In Appendix A we emphasize that the same results are valid in any Quasi-Stationary ROSt, not just the Boltzmann one.

2 Model, notations, previous basic results

The Hamiltonian of the SK model is defined on Ising spin configurations $\sigma : i \to \sigma_i = \pm 1$ of N spins, labeled by $i = 1, \ldots, N$, as

$$H_N(\sigma; J) = -\frac{1}{\sqrt{N}} \sum_{i<j}^{1,N} J_{ij} \sigma_i \sigma_j$$

where J_{ij} are i.i.d. centered unit Gaussian random variables. We will assume there is no external field. Being a centered Gaussian variable, the Hamiltonian is determined by its covariance

$$\mathbb{E}[H_N(\sigma) H_N(\sigma')] = \frac{1}{2} N q_{\sigma\sigma'}^2$$

where

$$q_{\sigma\sigma'} = \frac{1}{N} \sum_{i=1}^{N} \sigma_i \sigma_i'$$

is the overlap, and $\mathbb{E}$ denotes here the expectation with respect to all the (quenched) Gaussian variables.

The partition function $Z_N(\beta)$, the quenched free energy density $f_N(\beta)$ and pressure $\alpha_N(\beta)$ are defined as:

$$\begin{aligned} Z_N(\beta) &= \sum_{\sigma} \exp(-\beta H_N(\sigma)) \ , \\ -\beta f_N(\beta) &= \frac{1}{N} \mathbb{E} \ln Z_N(\beta) = \alpha_N(\beta) \ . \end{aligned}$$

The Boltzmann-Gibbs average of an observable $\mathcal{O}(\sigma)$ is denoted by ω and defined as

$$\omega(\mathcal{O}) = Z_N(\beta)^{-1} \sum_\sigma \mathcal{O}(\sigma) \exp(-\beta H_N(\sigma)) \ ,$$

but we will use the same ω to indicate in general (weighted) sums over spins or non-quenched variables, to be specified when needed, and with Ω we will mean the product (*replica*) measure of the needed number of copies of ω.

Let us now introduce an auxiliary system.

Definition 1 *A* Random Overlap Structure $\mathcal{R}$ *is a triple* $(\Sigma, \tilde{q}, \xi)$ *where*

- $\Sigma \ni \gamma$ *is a discrete space (set of abstract spin-configurations);*
- $\tilde{q} : \Sigma^2 \to [0,1]$ *is a positive definite kernel (*Overlap Kernel*), with* $|\tilde{q}| \leq 1$ *(and* $\tilde{q} = 1$ *on the diagonal of* Σ^2*);*
- $\xi : \Sigma \to \mathbb{R}_+$ *is a normalized discrete positive random measure, i.e. a system of random weights such that there is a probability measure* μ *on* $[0,1]^\Sigma$ *so that* $\sum_{\gamma \in \Sigma} \xi_\gamma < \infty$ *almost surely in the* μ*-sense.*

The randomness in the weights ξ is independent of the randomness of the quenched variables from the original system with spins σ. We equip a ROSt with two families of independent and centered Gaussians $\tilde{h}_i$ and $\hat{H}$ with covariances

$$\mathbb{E}[\tilde{h}_i(\gamma)\tilde{h}_j(\gamma')] = \delta_{ij}\tilde{q}_{\gamma\gamma'} \ , \tag{1}$$

$$\mathbb{E}[\hat{H}(\gamma)\hat{H}(\gamma')] = \tilde{q}^2_{\gamma\gamma'} \ . \tag{2}$$

Given a ROSt $\mathcal{R}$ we define the trial pressure as

$$G_N(\mathcal{R}) = \frac{1}{N}\mathbb{E}\ln \frac{\sum_{\sigma,\gamma}\xi_\gamma \exp(-\beta\sum_{i=1}^N \tilde{h}_i(\gamma)\sigma_i)}{\sum_\gamma \xi_\gamma \exp(-\beta\sqrt{\frac{N}{2}}\hat{H}(\gamma))} \ , \tag{3}$$

where $\mathbb{E}$ denotes hereafter the expectation with respect to all the (quenched) random variables (including the randomness in the random weights ξ) but spins σ and the abstract spins γ, the sum over which is in fact written explicitly.

The following theorem ([2]) can be easily proven by interpolation

Theorem 1 (Extended Variational Principle) *Infimizing for each N separately the trial function $G_N(\mathcal{R})$ defined in (3) over the whole ROSt space, the resulting sequence tends to the limiting pressure $-\beta f(\beta)$ of the SK model as N tends to infinity*

$$\alpha(\beta) \equiv \lim_{N\to\infty} \alpha_N(\beta) = \lim_{N\to\infty} \inf_{\mathcal{R}} G_N(\mathcal{R}) \ .$$

For a given ROSt, the trial pressures $\{G_N\}$ are a well defined sequence of real numbers indexed by N; a ROSt $\mathcal{R}$ is said to be optimal if $\alpha \equiv \lim_{N\to\infty}\alpha_N(\beta) = \lim_{N\to\infty} G_N(\mathcal{R}) \ \forall\beta$. See [4] for comments on some topological aspects of the ROSt theory. The space of all ROSt can be restricted to those ROSt's enjoying some factorization property that all optimal ROSt's enjoy, without missing the exact pressure [12, 10]. We will therefore limit ourselves to these ROSt's, called Quasi-Stationary.

An optimal ROSt is the Parisi one ([14, 16]), another optimal one is the so-called Boltzmann ROSt $\mathcal{R}_B$, defined as follows. Take $\Sigma = \{-1,1\}^M$, and denote by τ the points of Σ. We clearly have in mind an auxiliary spin systems (and that is why we use τ as opposed to the previous γ to denote

its points). In fact, we also choose

$$\tilde{h}_i = -\frac{1}{\sqrt{M}} \sum_{k=1}^{M} \tilde{J}_{ik} \tau_k \ , \quad \hat{H} = -\frac{1}{M} \sum_{k,l}^{1,M} \hat{J}_{kl} \tau_k \tau_l$$

which satisfy (1)-(2) with $\tilde{q}_{\tau\tau'} = \frac{1}{M} \sum_k \tau_k \tau'_k$, and $\tilde{J}$ and $\hat{J}$ are families of i.i.d. random variables independent of the original couplings J, with whom they share the same distribution (i.e. all the $\tilde{J}$ and $\hat{J}$ are centered unit Gaussian random variables). The variables $\tilde{h}_{\cdot}$ are called *cavity fields*. Let us also choose

$$\xi_\tau = \exp(-\beta H_M(\tau; \hat{J})) = \exp\left(\beta \frac{1}{\sqrt{M}} \sum_{k,l}^{1,M} \hat{J}_{kl} \tau_k \tau_l\right) .$$

If we call $\mathcal{R}_B(M)$ the structure defined above, we will formally write $\mathcal{R}_B(M) \to \mathcal{R}_B$ as $M \to \infty$, and we call $\mathcal{R}_B$ the Boltzmann ROSt. The reason why such a ROSt is optimal is purely thermodynamic, and equivalent to the existence of the thermodynamic limit of the free energy per spin. A detailed proof of this fact can be found in [2]; here we just mention the main point:

$$\alpha(\beta) = \mathbf{C} \lim_M \frac{1}{N} \mathbb{E} \ln \frac{Z_{N+M}}{Z_M} = \lim_{N \to \infty} \mathbf{C} \lim_M G_N(\mathcal{R}_B(M)) = G_N(\mathcal{R}_B) = G(\mathcal{R}_B)$$

where $\mathbf{C}\lim$ is the limit in the Cesàro sense. Notice that the Boltzmann ROSt does not depend on N, after the M-limit.

3 Analysis of the Boltzmann ROSt

In this section we show that in the optimal Boltzmann ROSt's the overlap fluctuations obey some restrictions, namely those found by Aizenman and Contucci in [1]. In other words we exhibit a recipe to generate the AC polynomials within the ROSt approach.

3.1 the internal energy term

Let us focus on the denominator of the trial pressure $G(\mathcal{R}_B)$, defined in (3), computed at the Boltzmann ROSt $\mathcal{R}_B$, defined in the previous section. Let us normalize this quantity by dividing by Z_N and weight $\hat{H}$ with an independent variable β' as opposed to β, which appears in the *Boltzmannfaktor* ξ_τ. As in the Boltzmann structure we have actual spins (τ) and we do not use the spins σ here, we will still use ω (or Ω) to denote the Boltzmann-Gibbs (replica) measure (at inverse temperature β) in the space $\Sigma = \{-1, 1\}^M$. Moreover, we will use the notation $\langle\cdot\rangle = \mathbb{E}\Omega(\cdot)$ and, if present, a subscript β recalls that the *Boltzmannfaktor* in Ω has inverse temperature β. More precisely, we are computing the left hand side of the next equality to get this

Lemma 1

$$\frac{1}{N}\mathbb{E}\ln\Omega\exp\left(-\beta'\sqrt{\frac{N}{2}}\hat{H}(\tau)\right) = \frac{\beta'^2}{4}(1 - \langle\tilde{q}^2\rangle_\beta) \ . \tag{4}$$

Similar calculations have been performed already, but in this specific context the result has been only stated without proof in [12], while a detailed proof is given only in the dilute case in [10]. So let us prove the lemma. Let us take M finite. Thanks to the property of addition of independent Gaussian variables, the left hand side of (4) is the same as

$$\frac{1}{N}\mathbb{E}\ln\frac{Z_M(\beta^*)}{Z_M(\beta)} = \frac{M}{N}(\alpha_M(\beta^*) - \alpha_M(\beta)) \ , \ \beta^* = \sqrt{\beta^2 + \frac{\beta'^2 N}{M}}$$

which in turn, thanks to the convexity of α, can be estimated as follows

$$\frac{M}{N}(\beta^* - \beta)\alpha'_M(\beta) \leq \frac{M}{N}(\alpha_M(\beta^*) - \alpha_M(\beta)) \leq \frac{M}{N}(\beta^* - \beta)\alpha'_M(\beta^*) \ .$$

Now

$$\frac{M}{N}(\beta^* - \beta) = \frac{\beta'^2}{2\beta} + o(\frac{1}{M}) \ , \ \alpha'(\beta) = \frac{\beta}{2}(1 - \langle \tilde{q}^2 \rangle_\beta) \ .$$

Therefore, when $M \to \infty$, we get (4) for almost all β, i.e. whenever $\alpha'(\beta^*) \to \alpha'(\beta)$, or equivalently whenever $\langle \cdot \rangle_{\beta^*} \to \langle \cdot \rangle_\beta$. Notice that the quantity in (4) does not depend on N [12, 10].

Theorem 2 *The following statements hold:*

- *The left hand side of (4) is intensive (does not depend on N);*
- *The left hand side of (4) is a monomial of order two in β';*
- *The Aizenman-Contucci identities hold.*

Proof

Recall that $\hat{H}$ is a centered Gaussian, and so is therefore $-\hat{H}$ and the Gibbs measure is such that the substitution $\hat{H} \to \hat{H} - \hat{H}'$ implies

$$\frac{1}{N}\mathbb{E}\ln\Omega\exp\left(-\beta'\sqrt{\frac{N}{2}}\hat{H}\right) = \frac{1}{2N}\mathbb{E}\ln\Omega\exp\left(-\beta'\sqrt{\frac{N}{2}}(\hat{H} - \hat{H}')\right) \ .$$

Expand now in powers of β' the exponential first and then the logarithm.

$$\begin{aligned}
\frac{1}{N}\mathbb{E}\ln\Omega\exp\left(-\beta'\sqrt{\frac{N}{2}}\hat{H}\right) &= \frac{\beta'^2}{4}(1 - \langle \tilde{q}^2 \rangle) = \\
\frac{1}{2N}\mathbb{E}\ln\Omega&\left[1 + \frac{\beta'^2}{2}\frac{N}{2}(\hat{H} - \hat{H}')^2 + \frac{\beta'^4}{4!}\frac{N^2}{4}(\hat{H} - \hat{H}')^4 + \cdots\right] = \\
&\frac{1}{2N}\mathbb{E}\left\{\left[\frac{N\beta'^2}{4}(2\Omega(\hat{H}^2) - 2\Omega^2(\hat{H}))\right] + \right. \\
&\frac{N^2}{4}\frac{\beta'^4}{4!}\left[2\Omega(\hat{H}^4) - 8\Omega(\hat{H})\Omega(\hat{H}^3) + 6\Omega^2(\hat{H}^2)\right] - \\
&\left.\frac{N^2}{2}\frac{\beta'^4}{4}\left[\Omega^2(\hat{H}^2) + \Omega^4(\hat{H}) - 2\Omega(\hat{H}^2)\Omega^2(\hat{H})\right] + \cdots\right\} \ .
\end{aligned}$$

A straightforward calculation yields

$$\mathbb{E}\Omega(\hat{H}^4) = 3 \ , \quad \mathbb{E}[\Omega(\hat{H}^3)\Omega(\hat{H})] = 3\langle \tilde{q}_{12}^2 \rangle \ , \quad \mathbb{E}\Omega^2(\hat{H}^2) = 1 + 2\langle \tilde{q}_{12}^4 \rangle \ ,$$

$$\mathbb{E}[\Omega(\hat{H}^2)\Omega^2(\hat{H})] = \langle \tilde{q}_{12}^2 \rangle + 2\langle \tilde{q}_{12}^2 \tilde{q}_{13}^2 \rangle \ , \quad \mathbb{E}\Omega^4(\hat{H}) = 3\langle \tilde{q}_{12}^2 \tilde{q}_{34}^2 \rangle$$

and so on. All quantities of this sort can be computed in the same way. As an example, let us calculate $\mathbb{E}[\Omega(\hat{H}^2)\Omega^2(\hat{H})] = \mathbb{E}[\omega(\hat{H}_1^2)\omega(\hat{H}_2)\omega(\hat{H}_3)]$. Like for overlaps, subscripts denote replicas. In order to evaluate the expectation of products of Gaussian variables, we can use Wick's theorem: we just count all the possible ways to contract the four Gaussian terms $\hat{H}_1, \hat{H}_1, \hat{H}_2, \hat{H}_3$ and sum over every non-vanishing contribution

$$\langle \hat{H}_1 \hat{H}_2 \hat{H}_1 \hat{H}_3 \rangle = \langle \tilde{q}_{12}^2 \tilde{q}_{23}^2 \rangle \ , \tag{5}$$

$$\langle \hat{H}_1 \hat{H}_1 \hat{H}_2 \hat{H}_3 \rangle = \langle 1 \cdot \tilde{q}_{12}^2 \rangle \ , \tag{6}$$

$$\langle \hat{H}_1 \hat{H}_3 \hat{H}_1 \hat{H}_2 \rangle = \langle \tilde{q}_{12}^2 \tilde{q}_{23}^2 \rangle \ . \tag{7}$$

The sum of all the terms gives the exactly $\langle \tilde{q}_{12}^2 \rangle + 2\langle \tilde{q}_{12}^2 \tilde{q}_{23}^2 \rangle$. Now equation (4) is therefore expressed in terms of an identity for all β' of two polynomials in β': one is of order two, the other is a whole power series. We can then equate the coefficients of same order, or equivalently put to zero all the terms of order higher that two in β'. The consequent equalities are exactly the Aizenman-Contucci ones ([1]), an example of these is

$$\langle \tilde{q}_{12}^4 \rangle - 4\langle \tilde{q}_{12}^2 \tilde{q}_{13}^2 \rangle + 3\langle \tilde{q}_{12}^2 \tilde{q}_{34}^2 \rangle = 0 \ ,$$

which arises from the lowest order in the expansion above. $\square$

3.2 the entropy term

In the same spirit as in the previous section, let us move on to the normalized numerator of the trial pressure $G(\mathcal{R}_B)$, defined in (3), computed at the Boltzmann ROSt $\mathcal{R}_B$, defined in the previous section. If we define

$$c_i = 2\cosh(-\beta\tilde{h}_i) = \sum_{\sigma_i} \exp(-\beta\tilde{h}_i\sigma_i) \ ,$$

then

$$\frac{1}{N}\mathbb{E}\ln\Omega\sum_{\sigma}\exp(-\beta\sum_{i=1}^{N}\tilde{h}_i\sigma_i) = \frac{1}{N}\mathbb{E}\ln\Omega(c_1\cdots c_N) \tag{8}$$

does not depend on N [12, 10], if we consider the infinite Boltzmann ROSt, where $M \to \infty$.

Again, assume we replace the β in front of the cavity fields $\tilde{h}_\cdot$ (but not in the state Ω) with a parameter $\sqrt{t}$, and define, upon rescaling,

$$\Psi(t) = \mathbb{E}\ln\Omega\sum_{\sigma}\exp\frac{\sqrt{t}}{\sqrt{N}}\sum_{i=1}^{N}\tilde{h}_i\sigma_i \ . \tag{9}$$

We want to study the flux (in t) of equation (9) to obtain an integrable expansion. The t-flux of the cavity function Ψ is given by

$$\partial_t\Psi(t) = \frac{1}{2}(1 - \langle q_{12}\tilde{q}_{12}\rangle_t) \ , \tag{10}$$

which is easily seen by means of a standard use of Gaussian integration by parts. The subscript in $\langle\cdot\rangle_t = \mathbb{E}\Omega_t$ means that such an average includes the t-dependent exponential appearing in (9), beyond the sum over σ.

Theorem 3 *Let F_s be measurable with respect to the σ-algebra generated by the overlaps of s replicas of $\{\sigma\}$ and $\{\tau\}$. Then the cavity streaming*

equation is

$$\partial_t\langle F_s\rangle_t =$$

$$\langle F_s(\sum_{\gamma,\delta}^{1,s} q_{\gamma,\delta}\tilde{q}_{\gamma,\delta} - s\sum_{\gamma=1}^{s} q_{\gamma,s+1}\tilde{q}_{\gamma,s+1} + \frac{s(s+1)}{2} q_{s+1,s+2}\tilde{q}_{s+1,s+2})\rangle_t \ . \quad (11)$$

Proof

We consider the Boltzmann ROSt $\mathcal{R}_B(M)$ with any value of M. The proof relies on the repeated application of the usual integration by parts formula for Gaussian variables

$$\begin{aligned}
\partial_t\langle F_s\rangle_t &= \partial_t\mathbb{E}\frac{\sum_{\sigma\tau} F_s \exp(-\beta H_M(\tau))\exp(\sqrt{\frac{t}{MN}}\sum_{ij}\sum_\gamma \tilde{J}_{ij}\tau_i^\gamma\sigma_j^\gamma)}{\sum_{\sigma\tau}\exp(-\beta H_M(\tau))\exp(\sqrt{\frac{t}{MN}}\sum_{ij}\sum_\gamma \tilde{J}_{ij}\tau_i^\gamma\sigma_j^\gamma)} \\
&= \frac{1}{2\sqrt{tMN}}\mathbb{E}\sum_{ij} J_{ij}\sum_{\gamma}^{s}(\Omega_t[F_s\tau_i^\gamma\sigma_j^\gamma] - \Omega_t[F_s]\Omega_t[\tau_i^\gamma\sigma_j^\gamma]) \\
&= \frac{1}{2\sqrt{tMN}}\sum_{ij}\mathbb{E}J_{ij}(\sum_\gamma \Omega_t[F_s\tau_i^\gamma\sigma_j^\gamma] - s\Omega_t[F_s]\omega[\tau_i\sigma_j]) \\
&= \frac{1}{2MN}\sum_{ij}\mathbb{E}(\sum_{\gamma,\delta}\Omega_t[F_s\sigma_j^\gamma\tau_i^\gamma\sigma_j^\delta\tau_i^\delta] \\
&\quad -\sum_{\gamma,\delta}\Omega_t[F_s\tau_i^\gamma\sigma_j^\gamma]\Omega_t[\tau_i^\delta\sigma_j^\delta] - s\omega_t[\tau_i\sigma_j]\sum_\delta(\Omega_t[F_s\tau_i^\delta\sigma_j^\delta] \\
&\quad -\Omega_t[F_s]\Omega_t[\tau_i^\delta\sigma_j^\delta] - s\Omega_t[F_s](1-\omega_t^2[\tau_i\sigma_j]))) \\
&= \frac{1}{2}\mathbb{E}(\sum_{\gamma,\delta}\Omega_t[F_s q_{\gamma,\delta}\tilde{q}_{\gamma,\delta}] - s\sum_\gamma \Omega_t[F_s q_{\gamma,s+1}\tilde{q}_{\gamma,s+1}] \\
&\quad +ss\Omega_t[F_s q_{s+1,s+2}\tilde{q}_{s+1,s+2}] \\
&\quad -s\Omega_t[F_s]\Omega_t[F_s q_{s+1,s+2}\tilde{q}_{s+1,s+2}]) \ ,
\end{aligned}$$

where in Ω_t we have included the sum over σ and τ, the *Boltzmannfaktor* in τ, and the t-dependent exponential. At this point, remembering that $\tilde{q}_{\gamma\gamma} = 1$, we can write

$$\sum_{\gamma,\delta}\Omega_t[F_s q_{\gamma\delta}\tilde{q}_{\gamma\delta}] = 2\sum_{\gamma,\delta}\Omega_t[F_s q_{\gamma\delta}\tilde{q}_{\gamma\delta}] + s\Omega_t[F_s] \ .$$

which completes the proof. □

Now the way to proceed is simple: we have to expand the t-derivative of $\Psi(t)$ (right hand side of (10)) using the cavity streaming equation (11), and we will stop the iteration at the first non trivial order (that is expected to be at least four, being the first AC relation of that order). Once a closed-form expression is in our hands, we can write down an order by order expansion of the (modified) denominator of the Boltzmann ROSt (that is the function $N^{-1}\psi(t)$ evaluated for $t = N\beta^2$). We have

$$\partial_t \langle q_{12}\tilde{q}_{12}\rangle_t = \langle q_{12}^2\tilde{q}_{12}^2 - 4q_{12}\tilde{q}_{12}q_{23}\tilde{q}_{23} + 3q_{12}\tilde{q}_{12}q_{34}\tilde{q}_{34}\rangle_t \ .$$

After the first iteration:

$$\partial_t \langle q_{12}^2\tilde{q}_{12}^2\rangle_t = \langle q_{12}^3\tilde{q}_{12}^3 - 4q_{12}^2\tilde{q}_{12}^2\tilde{q}_{23}p_{23} + 3q_{12}^2\tilde{q}_{12}^2\tilde{q}_{34}\rangle_t \ ,$$

$$\begin{aligned}\partial_t \langle \tilde{q}_{12}q_{12}\tilde{q}_{23}q_{23}\rangle_t &= \langle \tilde{q}_{12}q_{12}\tilde{q}_{23}q_{23}\tilde{q}_{13}q_{13} + 2\tilde{q}_{12}^2q_{12}^2\tilde{q}_{23}q_{23} \\ &- 6\tilde{q}_{12}q_{12}\tilde{q}_{23}q_{23}\tilde{q}_{34}q_{34} - 3\tilde{q}_{12}q_{12}\tilde{q}_{13}q_{13}\tilde{q}_{14}q_{14} + 6\tilde{q}_{12}q_{12}\tilde{q}_{34}q_{34}\tilde{q}_{45}q_{45}\rangle_t \ ,\end{aligned}$$

$$\begin{aligned}\partial_t \langle \tilde{q}_{12}q_{12}\tilde{q}_{34}q_{34}\rangle_t &= \langle 4\tilde{q}_{12}q_{12}\tilde{q}_{23}q_{23}\tilde{q}_{34}q_{34} + 2\tilde{q}_{12}^2q_{12}^2\tilde{q}_{34}q_{34} \\ &- 16\tilde{q}_{12}q_{12}\tilde{q}_{34}q_{34}\tilde{q}_{45}q_{45} + 10\tilde{q}_{12}q_{12}\tilde{q}_{34}q_{34}\tilde{q}_{56}q_{56}\rangle_t \ .\end{aligned}$$

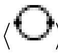
The higher orders can be obtained exactly in the same way, so we can write down right away the expression for $\langle q_{12}\tilde{q}_{12}\rangle$, referring to [1, 5] for a detailed explanation of this iterative method:

$$\begin{aligned}\langle q_{12}\tilde{q}_{12}\rangle_t &= \langle q_{12}^2\tilde{q}_{12}^2\rangle t - 2\langle q_{12}\tilde{q}_{12}q_{23}\tilde{q}_{23}q_{13}\tilde{q}_{13}\rangle t^2 - \frac{1}{6}\langle q_{12}^4\tilde{q}_{12}^4\rangle t^3 \\ &- 2\langle q_{12}^2\tilde{q}_{12}^2q_{23}^2\tilde{q}_{23}^2\rangle t^3 + \frac{3}{2}\langle q_{12}^2\tilde{q}_{12}^2q_{34}^2\tilde{q}_{34}^2\rangle t^3 + 6\langle q_{12}\tilde{q}_{12}q_{23}\tilde{q}_{23}q_{34}\tilde{q}_{34}q_{14}\tilde{q}_{14}\rangle t^3 \ .\end{aligned} \tag{12}$$

Notice that the averages no longer depend on t. In this expansion we considered both q-overlaps and $\tilde{q}$-overlaps, but as the sum over the spins σ can be performed explicitly, we can obtain an explicit expression at least for the q-overlaps, and get

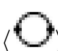
$$
\begin{aligned}
\langle q_{12}^2 \rangle &= \frac{1}{N^2}\mathbb{E}\sum_{ij}\omega^2(\sigma_i\sigma_j) = \frac{1}{N} \ , \\
\langle q_{12}q_{23}q_{31} \rangle &= \frac{1}{N^3}\mathbb{E}\sum_{ijk}\omega(\sigma_i\sigma_j)\omega(\sigma_j\sigma_k)\omega(\sigma_k\sigma_i) = \frac{1}{N^2} \ , \\
\langle q_{12}^2 q_{34}^2 \rangle &= \frac{1}{N^4}\mathbb{E}\sum_{ijkl}\omega^2(\sigma_i\sigma_j)\omega^2(\sigma_k\sigma_l) = \frac{1}{N^2} \ , \\
\langle q_{12}q_{23}q_{34}q_{14} \rangle &= \frac{1}{N^4}\mathbb{E}\sum_{ijkl}\omega(\sigma_i\sigma_j)\omega(\sigma_j\sigma_k)\omega(\sigma_k\sigma_l)\omega(\sigma_l\sigma_i) = \frac{1}{N^3} \ , \\
\langle q_{12}^4 \rangle &= \frac{1}{N^4}\mathbb{E}\sum_{ijkl}\omega(\sigma_i\sigma_j\sigma_k\sigma_l)\omega(\sigma_i\sigma_j\sigma_k\sigma_l) = \frac{3(N-1)}{N^3} + \frac{1}{N^3} \ , \\
\langle q_{12}^2 q_{23}^2 \rangle &= \frac{1}{N^4}\mathbb{E}\sum_{ijkl}\omega(\sigma_i\sigma_j)\omega(\sigma_i\sigma_j\sigma_k\sigma_l)\omega(\sigma_i\sigma_j) = \frac{1}{N^2} \ .
\end{aligned}
$$

Moreover, as the q-overlaps have been calculated explicitly, we can use a graphical formalism [1, 5]. In such a formalism we use points to identify replicas and lines for the overlaps between them. So for example:

⟨⟩ $= \langle \tilde{q}_{12} \rangle$, ⟨⟩ $= \langle \tilde{q}_{12}^2 \rangle$, ⟨⟩ $= \langle \tilde{q}_{12}\tilde{q}_{23}\tilde{q}_{13} \rangle$

and so on. Now we can integrate (10) thanks to the polynomial expansion based on (12) and to the expressions for the q-fluctuations. We obtain

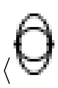
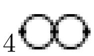
$$
\Psi(t) = \frac{1}{2}\int_0^t [1 - \langle q_{12}\tilde{q}_{12} \rangle_{t'}]dt' \ ,
$$

$\frac{1}{N}\Psi(t = N\beta^2) = \frac{\beta^2}{2} - \langle$$\rangle\frac{\beta^4}{4} + \langle$$\rangle\frac{\beta^6}{3} - \langle$$\rangle\frac{\beta^8}{24} -$

$\langle$$\rangle\frac{3\beta^8}{4} + N\beta^8[\frac{1}{16}\langle$$\rangle - \frac{1}{4}\langle$$\rangle + \frac{3}{16}\langle$$\rangle]$. (13)

This expression, though truncated at this low order, already looks pretty much alike the expansion found using the internal energy part of the Boltzmann pressure.

We stress however two important features of expression (13). The first is that within this approach we do not have problems concerning the Replica Simmetry Anzatz (RS) [14], and this can be seen by the proliferating of the overalaps fluctuations, via which we expand the entropy (a RS theory does not allow such fluctuations). Secondly, we note that not all the terms inside the equations (13) are intensive: the last three graphs are all multiplied by a factor N. Recalling that this expansion does not depend on N, and physically a density is intensive by definition, we put to zero all the terms in the squared bracket, so to have

$\langle$  $- 4$  $+ 3$  $\rangle = 0$.

Again we can find the AC identities.

A Extension to all Quasi-Stationary ROSt's

For sake of simplicity, all the explicit calculation we performed took into account the Boltmann structure only. But the whole content actually does not depend on the explicit form of the Hamiltonians, it merely relies on the Gaussian nature of the random variables and their moments, independently of the space they are defined in. In other words, as long as we consider centered Gaussian variables, the whole treatment depends only on their covariances. That is why changing the ROSt does not change the results, except the overlaps in the various expressions will be those of the considered

ROSt (e.g. the ultrametric Parisi trial overlaps), provided some properties are preserved (Quasi-Stationarity).

Let us focus for instance on the internal energy part, which is simpler, and see that the results of subsection 3.1 are the same in any Quasi-Stationary ROSt. First of all, notice that the proof of Theorem 2 never makes use of the explicit form of the Hamiltonians and therefore (5)-(6)-(7) stay identical, as they are determined purely by the covariances of the Hamiltonians. The same clearly holds for all the other terms not explicitly considered in the example.

So the validity of the results coincides with the validity of Lemma 1. The ROSt's for which such a lemma holds are called Quasi-Stationary (see [3, 4]), in this case with respect to the Cavity Step (see [12, 3]). Notice that the left hand side of (4) is zero for $\beta' = 0$ independently of the particular ROSt. Hence by the fundamental theorem of calculus the same left hand side coincides with the integral from zero to β' of its derivative (with respect to β'). But the form of such a derivative is just determined by the covariance of $\hat{H}$ (this is at the heart of [2]), which is always defined to be an overlap. Therefore a simple Gaussian integration by parts, as illustrated in [2], leads to the right hand side of (4). These are the intuitive reasons that heuristically explain why both the explicit calculation in Lemma 1 and the expansion of Theorem 2 are the same in any Quasi-Stationary ROSt, and so are the AC polynomials, no matter what the overlap looks like in a generic abstract space. So AC polynomials hold in any Quasi-Stationary ROSt, non-optimal too, but if the chosen ROSt is not optimal there will be no overlap locking and the trial overlap will have very little to share

with the true ones of the model. Moreover (4) will not in general provide the internal energy of the model (but this can be the case in some optimal ROSt too, like the Parisi one!).

Conclusions and Outlook

We have shown how some constraints on the distribution of the overlap naturally arise within the Random Overlap Structure approach. As our analysis of the Boltzmann ROSt is similar to the study of stochastic stability it is not surprising that the constraints coincide with the Aizenman-Contucci identities. In the ROSt context, such identities are easily connected with the existence of the thermodynamic limit of the free energy density (which is equivalent to the optimality of the Boltzmann ROSt) and with the physical fact that the internal energy is intensive. We also showed that, as expected, the entropy part of the free energy yields the same constraints as the other part (i.e. the internal energy).

The hope for the near future is that the ROSt approach will lead eventually to a good understanding of the pure states and the phase transitions of the model. A first step has been taken in [12], and our present results can be considered as a second step in this direction. (Other more interesting results regarding the phase transition at $\beta = 1$ can also be obtained with the same techniques employed here, including the graphical representation [6].) A further step should bring the Ghirlanda-Guerra identities, and then hopefully a proof of ultrametricity.

Acknowledgment

The authors warmly thank Francesco Guerra and Pierluigi Contucci for a precious scientific and personal support, and sincerely thank Peter Sollich for useful discussions and much advice. A grateful appreciation is owed to Louis-Pierre Arguin for pointing out a weak point in a previous version of the paper and for very fruitful conversations.

References

[1] M. Aizenman, P. Contucci, *On the stability of the quenched state in mean field spin glass models*, J. Stat. Phys. **92**, 765-783 (1998).

[2] M. Aizenman, R. Sims, S. L. Starr, *An Extended Variational Principle for the SK Spin-Glass Model*, Phys. Rev. B, **68**, 214403 (2003).

[3] M. Aizenman, R. Sims, S. L. Starr, *Mean-Field Spin Glass models from the Cavity-ROSt Perspective*, ArXiv: `math-ph/0607060`.

[4] L.-P. Arguin, *pin Glass Computations and Ruelle's Probability Cascades*, ArXiv: `math-ph/0608045`, to appear in J. Stat. Phys..

[5] A. Barra, *Irreducible free energy expansion for mean field spin glass model*, J. Stat. Phys. (2006) DOI: 10.1007/s10955-005-9006-6.

[6] A. Agostini, A. Barra, L. De Sanctis, in preparation.

[7] A. Bovier, *Statistical Mechanics of Disordered Systems, A mathematical perspective*, Cambridge University Press (2006).

[8] P. Contucci, C. Giardinà, *Spin-Glass Stochastic Stability: a Rigorous Proof*, ArXiv: `math-ph/0408002`.

[9] P. Contucci, C. Giardinà, *The Ghirlanda-Guerra identities*, ArXiv: `math-ph/0505055v1`.

[10] L. De Sanctis, *Random Multi-Overlap Structures and Cavity Fields in Diluted Spin Glasses*, J. Stat. Phys. **117** 785-799 (2004).

[11] S. Ghirlanda, F. Guerra, *General properties of overlap distributions in disordered spin systems. Towards Parisi ultrametricity*, J. Phys. A, **31** 9149-9155 (1998).

[12] F. Guerra, *About the Cavity Fields in Mean Field Spin Glass Models*, ArXiv: `cond-mat/0307673`.

[13] F. Guerra, *Broken Replica Symmetry Bounds in the Mean Field Spin Glass Model*, Commun, Math. Phys. **233:1** 1-12 (2003).

[14] M. Mézard, G. Parisi and M. A. Virasoro, *Spin glass theory and beyond*, World Scientific, Singapore (1987).

[15] G. Parisi, *On the probabilistic formulation of the replica approach to spin glasses*, Int. Jou. Mod. Phys. B **18**, 733-744, (2004).

[16] M. Talagrand, *The Parisi Formula*, Annals of Mathematics, 163, 221 (2006).

[17] M. Talagrand, *Spin glasses: a challenge for mathematicians. Cavity and Mean field models*, Springer Verlag (2003).